The Girsanov Multiplier

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May 2017

In probability theory, the **Girsanov theorem** (named after Igor Vladimirovich Girsanov) describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure. The theorem is especially important in the theory of financial mathematics as it tells how to convert from the physical measure, which describes the probability that an underlying instrument (such as a share price or interest rate) will take a particular value or values, to the risk-neutral measure, which is a very useful tool for pricing derivatives on the underlying. [1]

In this white paper we will use the Girsanov multiplier to move the mean of the actual probability distribution to the mean of the risk-neutral probability distribution. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with calculating the price of a option given the following assumptions...

Table 1: Model Assumptions

Symbol	Description	Value
A_0	Asset price at time zero	1,000
α	Risk-free interest rate (%)	4.00
μ	Cash flow growth rate (%)	3.50
κ	Cost of capital (%)	13.50
σ	Annual return volatility (%)	30.00

To price the option we need to determine the parameters for the risk-neutral probability distribution. We will use our model to answer the following question:

Question: What is the mean and variance of the risk-neutral probability distribution at time $t = 3.00$?

Asset Price Equation

We will define the variable A_t to be asset price at time t, the variable κ to be the cost of capital, the variable ϕ to be the dividend yield, the variable σ to be return volatility, and the variable W_t to be a Brownian motion. The stochastic differential equation that defines the change in asset price over time is...

$$\delta A_t = (\kappa - \phi)\, A_t\, \delta t + \sigma\, A_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

The solution to Equation (1) above is the equation for asset price at time t, which is...

$$A_t = A_0 \operatorname{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, Z\right\} \text{ ...where... } Z \sim N\Big[0, 1\Big] \tag{2}$$

We will define **Measure P** to be the actual probability measure. Using Equation (2) above it can be shown that the expected asset price at time t under the actual probability measure is the following equation...

$$\mathbb{E}^P\Big[A_t\Big] = \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\pi}} \operatorname{Exp}\left\{-\frac{1}{2}Z^2\right\} A_0 \operatorname{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, Z\right\}\delta Z = A_0 \operatorname{Exp}\left\{\left(\kappa - \phi\right)t\right\} \tag{3}$$

We will define **Measure Q** to be the risk-neutral probability measure and the variable α to be the risk-free rate. Under the risk-neutral measure all assets have a total return equal to the risk-free rate. Expected total return consists of capital gains (increase in asset price) plus dividend income. Since total return under the risk-neutral measure must equal the risk-free rate then the equation for total return and the components of total return must be...

$$\text{Dividends} = \phi\, A_t\, \delta t \;\Big|\; \text{Capital gains} = (\alpha - \phi)\; A_t\, \delta t \;\Big|\; \text{Total return} = \phi\, A_t\, \delta t + (\alpha - \phi)\; A_t\, \delta t = \alpha\, A_t\, \delta t \tag{4}$$

Expected asset price at time t consists of asset price at time zero plus capital gains. Using Equation (4) above expected asset price at time t under the risk-neutral measure is...

$$\mathbb{E}^Q\Bigg[A_t\Bigg] = A_0 \,\text{Exp}\Bigg\{(\alpha - \phi)\, t\Bigg\} \tag{5}$$

The Girsanov Multiplier

We will define the function $g(Z)$ to be the Girsanov multiplier and the variable Z to be a normally-distributed random variable with m and variance v. The equation for the function $g(Z)$ that moves the mean of the distribution from m to n is...

$$g(Z) = \text{Exp}\left\{\frac{n-m}{v}\, Z - \frac{n^2 - m^2}{2\, v}\right\} \;\text{...where...}\; Z \sim N\Bigg[m, v\Bigg] \;\text{...and...}\; n = \text{new mean} \tag{6}$$

The probability density function gives us the height of the normal curve. We will define the function $p(Z)$ to be the probability density function for a normal distribution with mean m and variance v. The equation for the probability density function is...

$$p(Z) = \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z - m\right)^2\right\} \tag{7}$$

We will define the function $q(Z)$ to be the probability density function for a normal distribution with mean n and variance v. The equation for the probability density function for this distribution is...

$$q(Z) = \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z - n\right)^2\right\} \tag{8}$$

Note that the variance of $q(Z)$ is the same as the variance of $p(Z)$ but the mean has changed from m to n. We want to use the Girsanov multipler to move the mean of $p(Z)$ to the mean of $q(Z)$ but keep the variance the same. Using Equations (6), (7) and (8) above we want to prove the following equation...

$$p(Z)\, g(Z) = q(Z) \tag{9}$$

The solution to Equation (9) above is...

$$\begin{aligned}
p(Z)\, g(Z) &= \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z - m\right)^2\right\}\text{Exp}\left\{\frac{n-m}{v}\, Z - \frac{n^2 - m^2}{2\, v}\right\} \\
&= \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z^2 - 2\, m\, Z + m^2\right)\right\}\text{Exp}\left\{\frac{n}{v}\, Z - \frac{m}{v}\, Z - \frac{n^2}{2\, v} + \frac{m^2}{2\, v}\right\} \\
&= \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z^2 - 2\, m\, Z + m^2 - 2\, n\, Z + 2\, m\, Z + n^2 - m^2\right\}\right. \\
&= \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z^2 - 2\, n\, Z + n^2\right\}\right. \\
&= \sqrt{\frac{1}{2\,\pi\, v}}\,\text{Exp}\left\{-\frac{1}{2\, v}\left(Z - n\right)^2\right\}
\end{aligned} \tag{10}$$

Since the end result of Equation (10) equals Equation (8) above then Equation (9) above is proved.

The Risk-Neutral Probability Distribution

Using Equations (2), (6) and (7) above we can rewrite Equation (5) above as...

$$\mathbb{E}^Q\left[A_t\right]=\int\limits_{-\infty}^{\infty}p(Z)\,g(Z)\,A_t\,\delta Z=A_0\,\mathrm{Exp}\left\{(\alpha-\phi)\,t\right\} \tag{11}$$

Using Equations (2) and (7) above the equation for $p(Z)$ in Equation (11) above is...

$$p(Z)=\sqrt{\frac{1}{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}\left(Z-m\right)^2\right\}=\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\ \text{...because...}\ m=0\ \text{...and...}\ v=1 \tag{12}$$

Using Equations (2) and (6) above the equation for $g(Z)$ in Equation (11) above is...

$$g(Z)=\mathrm{Exp}\left\{\frac{(n-m)}{v}\,Z-\frac{n^2-m^2}{2\,v}\right\}=\mathrm{Exp}\left\{n\,Z-\frac{1}{2}\,n^2\right\}\ \text{...because...}\ m=0\ \text{...and...}\ v=1 \tag{13}$$

Using Equations (2), (12) and (13) above we can rewrite Equation (11) above as...

$$A_0\,\mathrm{Exp}\left\{(\alpha-\phi)\,t\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t+\sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{n\,Z-\frac{1}{2}\,n^2\right\}\delta Z$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\mathrm{Exp}\left\{\sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{n\,Z-\frac{1}{2}\,n^2\right\}\delta Z$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,(\sigma\sqrt{t}+n)\,Z+n^2\right)\right\}\delta Z \tag{14}$$

We will define the variable θ as follows...

$$\theta=Z-(\sigma\sqrt{t}+n)\ \text{...where...}\ \theta^2=Z^2-2\,(\sigma\sqrt{t}+n)\,Z+n^2+2\,n\,\sigma\sqrt{t}+\sigma^2 t \tag{15}$$

Using Equation (15) above we can rewrite Equation (14) above as...

$$A_0\,\mathrm{Exp}\left\{(\alpha-\phi)\,t\right\}=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(\theta^2-2\,n\,\sigma\sqrt{t}-\sigma^2 t\right)\right\}\delta Z$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\mathrm{Exp}\left\{n\,\sigma\sqrt{t}+\frac{1}{2}\,\sigma^2 t\right\}\delta Z$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi\right)t+n\,\sigma\sqrt{t}\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z \tag{16}$$

Note that the derivative of theta in Equation (15) above with respect to the random variable Z is...

$$\frac{\delta\theta}{\delta Z}=1\ \text{...such that...}\ \delta Z=\delta\theta \tag{17}$$

Using Equation (17) above we can rewrite Equation (16) above as...

$$A_0\,\mathrm{Exp}\left\{(\alpha-\phi)\,t\right\}=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi\right)t+n\,\sigma\sqrt{t}\right\}\int\limits_{-\infty-(\sigma\sqrt{t}+n)}^{\infty-(\sigma\sqrt{t}+n)}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi\right)t+n\,\sigma\sqrt{t}\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta$$

$$=A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi\right)t+n\,\sigma\sqrt{t}\right\}\ \text{...because...}\ \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta=1 \tag{18}$$

Taking the log of Equation (18) above and solving for n we get...

$$\ln(A_0) + (\alpha - \phi)\,t = \ln(A_0) + (\kappa - \phi)\,t + n\,\sigma\sqrt{t}$$
$$(\alpha - \phi)\,t = (\kappa - \phi)\,t + n\,\sigma\sqrt{t}$$
$$(\alpha - \kappa)\,t \div \sigma\sqrt{t} = n \tag{19}$$

The Solution To Our Hypothetical Problem

Using Table 1 above the continuous time equivalents of the cash flow growth rate, cost of capital and dividend yield are...

$$\mu = \ln\left(1 + 0.0350\right) = 0.0344 \text{ ...and.... } \kappa = \ln\left(1 + 0.1350\right) = 0.1266 \text{ ...and.... } \phi = \kappa - \mu = 0.0922 \tag{20}$$

Using Table 1 above the continuous time equivalent of the risk-free rate is...

$$\alpha = \ln\left(1 + 0.0400\right) = 0.0392 \tag{21}$$

Question: What is the mean and variance of the risk-neutral probability distribution at $t = 3.00$?

Answer: Using Equations (19), (20) and (21) above and the parameters from Table 1 above the answer to our hypothetical equations is...

$$n = \left(\alpha - \kappa\right)t \Big/ \sigma\sqrt{t} = \left(0.0392 - 0.1266\right) \times 3.00 \Big/ \left(0.3000 \times \sqrt{3.00}\right) = -0.5047 \tag{22}$$

References

[1] Wikipedia definition of the Girsanov theorem.